Throughout $D \subseteq \mathbb{R}$, x_0 cluster w.r.t to D

& $g, f, f_i : D \to \mathbb{R}$, $l, l_1, l_2 \in \mathbb{R}$. $k \in \mathbb{R}, k > 0$.

Definition of Limits & equivalent ones

① use of $\leq$ in place of $<$ (regarding ε, δ)

② use $k\varepsilon$, in place of ε,
$k\delta$ — — — δ etc.

③ limit description is "local" : if $\exists\, \gamma > 0$ s.t.

$$f_1(x) = f_2(x) \quad \forall x \in V_\gamma(x_0) \cap (D \setminus \{x_0\})$$

then $f_1(x) \to l$ as $x \to x_0$ iff $f_2(x) \to l$ as $x \to x_0$

④ some easy rules : multiplication by $k \in \mathbb{R}$ (pos or not)
addition / subtraction
absolute values
squeeze ; $\left\{ \begin{array}{l} 0 \leq f(x) \leq g(x) \ \forall x \in D \\ \lim_{x \to x_0} g(x) = 0 \end{array} \right.$

$$\Rightarrow \quad \lim_{x \to x_0} f(x) = 0.$$

⑤ deeper results : order-preserving
boundedness

⑥ More on computation rules $\left\{ \begin{array}{l} \text{product} \\ \text{quotient} \\ \text{sq. roots} \end{array} \right.$

<u>proof of product (computation) rule</u>. Let $\lim_{x \to x_0} f_i(x) = l_i$. Then, by Boundedness Th, $\exists M > 0$ & $\delta_i > 0$ s.t.

(1) $|f_i(x)| \leq M \quad \forall x \in (V_{\delta_i}(x_0) \setminus \{x_0\}) \cap D, \quad i = 1, 2.$

Replaced by $M + |l_1| + |l_2|$ if necessary, we may assume furthermore that $|l_1|, |l_2| \leq M$.

Let $\varepsilon > 0$. Let $\varepsilon' > 0$ be s.t.

$$\varepsilon' \leq \frac{\varepsilon}{2M}$$

For this $\varepsilon' > 0$, $\exists \ \delta_i' > 0$ s.t.

(2) $|f_i(x) - l_i| < \varepsilon' \quad \forall x \in (V_{\delta_i'}(x_0) \setminus \{x_0\}) \cap D \quad (i = 1, 2)$

Let $\delta = \min\{\delta_1, \delta_2, \delta_1', \delta_2'\}$. Then, applying (1) and (2) one has $\forall x \in V_\delta$

$$|f_1(x) f_2(x) - l_1 l_2| \leq |f_1(x) f_2(x) - f_1(x) l_2 + f_1(x) l_2 - l_1 l_2|$$

$$\leq M |f_2(x) - l_2| + M |f_1(x) - l_1|$$

$$< 2M\varepsilon' \leq \varepsilon .$$

QED.

Proof of Max-Rule. Let $\lim_{x \to x_0} f_i(x) = l_i$ and $l = l_1 \vee l_2$. We do our verification in two separate cases: (1) l_1, l_2 are distinct (say $l_1 < l_2$); (2) $l_1 = l_2$.

For case (1), one has $l = l_2$. Let $\gamma = \frac{l_1 + l_2}{2}$; then

$$\lim_{x \to x_0} f_1(x) = l_1 < \gamma < l_2 = \lim_{x \to x_0} f_2(x).$$

By the order-preserving property, $\exists\, \delta_i > 0$ s.t.

$$f_1(x) < \gamma \qquad \forall\, x \in \left(V_{\delta_1}(x_0) \setminus \{x_0\}\right) \cap D$$

and

$$\gamma < f_2(x) \quad \forall\, x \in \left(V_{\delta_2}(x_0) \setminus \{x_0\}\right) \cap D;$$

hence, with $\delta := \delta_1 \wedge \delta_2$, one has

$$f_1(x) < \gamma < f_2(x) \quad \forall\, x \in \left(V_\delta(x_0) \setminus \{x_0\}\right) \cap D$$

so

$$f_1(x) \vee f_2(x) = f_2(x) \quad \forall\, x \in \text{------}$$

Consequently

$$\lim_{x \to x_0} \left(f_1(x) \vee f_2(x)\right) = \lim_{x \to x_0} f_2(x) = l_2 \left(= l_1 \vee l_2 = l\right).$$

For case (2) $(l_1 = l_2)$ one has $l = l_1 = l_2$. Let $\varepsilon > 0$.
Then $\exists\, \delta_1, \delta_2 > 0$ s.t.

$$l_i - \varepsilon < f_i(x) < l_i + \varepsilon \quad \forall\, x \in \left(V_{\delta_i}(x_0) \setminus \{x_0\}\right) \cap D$$

Letting $\delta = \delta_1 \wedge \delta_2$ and noting $l_1 = l_2 = l$ one has

$$l - \varepsilon < \begin{Bmatrix} f_1(x) \\ f_2(x) \end{Bmatrix} < l + \varepsilon \quad \forall x \in \left(V_\delta(x_0) \setminus \{x_0\}\right) \cap D$$

and so

$$l - \varepsilon < f_1(x) \vee f_2(x) < l + \varepsilon \quad \forall x \in (V_\delta(x_0) \setminus \{x_0\}) \cap D.$$

This shows that $\lim_{x \to x_0} (f_1(x) \vee f_2(x)) = l$.

Remark. Suppose $\lim_{x \to x_0} f(x) = L$. Then one case show that $\lim_{x \to x_0} |f(x)| = |L|$ by $\varepsilon - \delta$ terminology, or via

$$|f(x)| = f^+(x) + f^-(x) \quad \forall x \in D,$$

where, $\forall\, r \in \mathbb{R}$,

$$r^+ := \max\{r, 0\}$$
$$r^- := \max\{-r, 0\}.$$

Another approach: Check $\lim_{x \to x_0} |f(x)| = |L|$ when it is given that $\lim_{x \to x_0} f(x) = L$. Then one case establish the max-rule via

$$\lim_{x \to x_0} (f_1(x) \vee f_2(x)) = \lim_{x \to x_0} \left[\frac{f_1(x) + f_2(x) + |f_1(x) - f_2(x)|}{2} \right]$$